Trinity College

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 1

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures | | | | | | | | |

In words SOLN'S

Your name

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	~~49~~ 52	35
Section Two: Calculator-assumed	13	13	100	102	65
			Total	~~151~~ 154	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

See next page

Section One: Calculator-free **35% (~~49~~ 52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(7 marks)**

(a) Evaluate

(i) $\frac{8!}{2!3!4!}$. $= \frac{8.7.6.5}{2.3.2.} = 140$ (2 marks)

(ii) $\frac{{}^{20}P_6}{{}^{21}C_{14}}$. $= \frac{20!}{14!} \times \frac{14!\ 7!}{21!}$ (3 marks)

$= 6.5.4.2$

$= 240$

(b) Determine the values of a and b given $8!+9!+10!=a\times b!$ (2 marks)

$8!+9!+10!$

$= 8!\,(1+9+90)$

$= 100 \times 8!$ $\Rightarrow$ $a = 100$, $b = 8$

Question 2 **(8 marks)**

Given $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$, determine

(a) $5\mathbf{a} + 10\mathbf{b}$. (1 mark)

$= \begin{pmatrix} 10 \\ -25 \end{pmatrix} + \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 20 \\ -15 \end{pmatrix}$

(b) $4(\mathbf{b} - 2\mathbf{a})$. (1 mark)

$= 4\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -10 \end{pmatrix}\right] = 4\begin{pmatrix} -3 \\ 11 \end{pmatrix} = \begin{pmatrix} -12 \\ 44 \end{pmatrix}$

(c) $|\mathbf{a} + 6\mathbf{b}|$. (2 marks)

$\underset{\sim}{a} + 6\underset{\sim}{b} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

$|\underset{\sim}{a} + 6\underset{\sim}{b}| = \sqrt{64 + 1}$

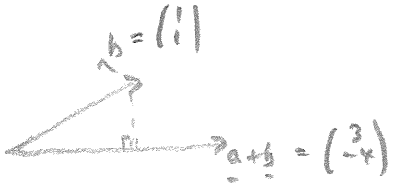
$= \sqrt{65}$

(d) a unit vector in the same direction as $\mathbf{a} + \mathbf{b}$. (2 marks)

$\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \qquad |\underset{\sim}{a} + \underset{\sim}{b}| = 5$

$\therefore \widehat{(\underset{\sim}{a} + \underset{\sim}{b})} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

(e) the scalar projection of $\mathbf{b}$ onto $\mathbf{a} + \mathbf{b}$. (2 marks)



i.e. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$= \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}}{5}$

$= -\frac{1}{5}$

See next page

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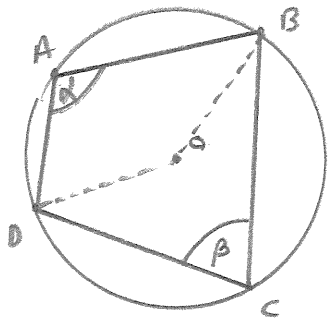
Question 3 **(7 marks)**

(a) Prove that the opposite angles of a cyclic quadrilateral are supplementary. (4 marks)



Let $\angle BAD = \alpha$ and $\angle BCD = \beta$ join BO & DO.

on arc BAD $\angle BOD = 2\beta$

on arc BCD $\angle BOD = 2\alpha$ (reflex angle)

now $2\alpha + 2\beta = 360°$

ie $\alpha + \beta = 180°$ Q.E.D.

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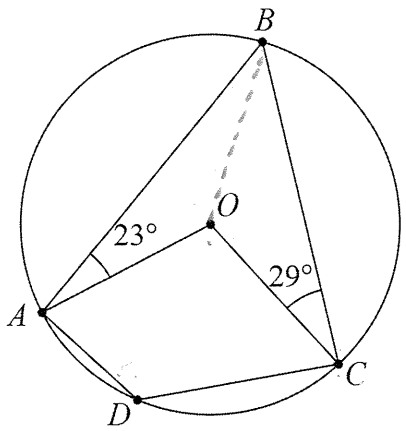
(b) Determine, with reasons, the size of $\angle ADC$ in the diagram below. (3 marks)



$\angle COB = 122°$ (isosceles Δ) $180 - 2(29)$

$\angle AOB = 134$ (") $180 - 2(23)$

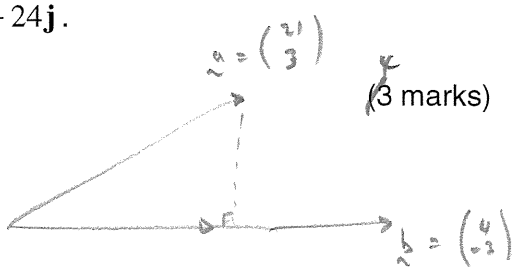
$\therefore \angle AOC = 256°$ (reflex)

$\therefore \angle ADC = 128°$ (angle at centre + circumference for arc ABC)

Question 4 **(7 marks)**

Consider the vectors $\mathbf{a} = 21\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = 18\mathbf{i} + 24\mathbf{j}$.

(a) Determine the vector projection of $\mathbf{a}$ onto $\mathbf{b}$. (3 marks)



$\hat{b} = \frac{1}{5}\begin{pmatrix}4\\-3\end{pmatrix}$ now $\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}\cdot\hat{\underset{\sim}{b}}$

$= \frac{\begin{pmatrix}21\\3\end{pmatrix}\cdot\begin{pmatrix}4\\-3\end{pmatrix}}{5.}\cdot\frac{\begin{pmatrix}4\\-3\end{pmatrix}}{5}$

$= \frac{75}{25}\begin{pmatrix}4\\-3\end{pmatrix} = 3\begin{pmatrix}4\\-3\end{pmatrix}$ $\left[=\begin{pmatrix}12\\-9\end{pmatrix}\right]$

(b) Express $\mathbf{c}$ in the form $x\mathbf{a} + y\mathbf{b}$. (4 marks)

$x\begin{pmatrix}21\\3\end{pmatrix} + y\begin{pmatrix}4\\-3\end{pmatrix} = \begin{pmatrix}18\\24\end{pmatrix}$

ie $21x + 4y = 18$ — ①

$3x - 3y = 24 \Rightarrow y = x - 8$ in ① & get

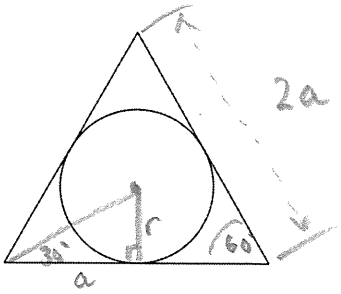
$21x + 4x - 32 = 18$

$25x = 50 \Rightarrow x = 2$ $\therefore y = -6$

$\therefore \underset{\sim}{c} = 2\underset{\sim}{a} - 6\underset{\sim}{b}$

Question 5 **(8 marks)**

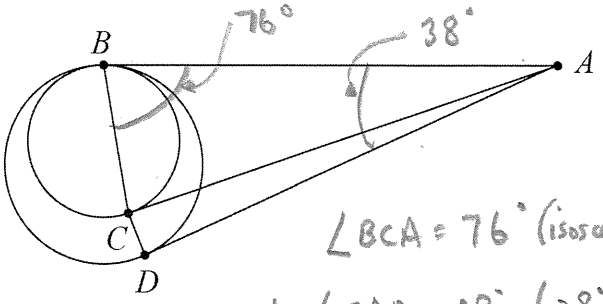
(a) An equilateral triangle of side $2a$ circumscribes a circle, as shown in the diagram below. Express the exact radius of the circle in terms of a. (4 marks)



$\tan 30° = \frac{r}{a}$

$\therefore r = \frac{a}{\sqrt{3}}$ or $\frac{a\sqrt{3}}{3}$

(b) Two circles touch internally at B, as shown below. AB, AC and AD are tangents, $\angle ABC = 76°$ and $\angle BAD = 38°$. Determine, with reasons, the size of $\angle CDA$. (4 marks)



$\angle BCA = 76°$ (isosceles $\triangle$)

$\therefore \angle CAD = 38° - (28°)$

$= 10°$

$\therefore \angle CDA = \frac{180° - 10°}{2}$

$= 85°$

$\triangle BAC$ is isosceles (tangents from A to ⊙ are equal)

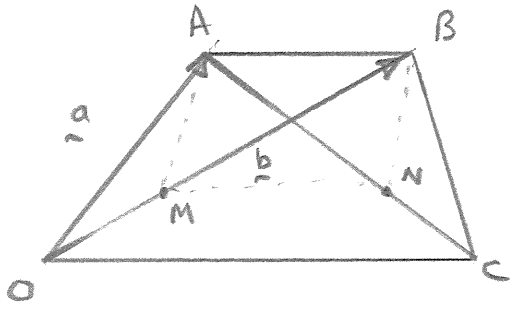
$\triangle BAD$ is isosceles (ditto as before)

$\therefore \triangle CAD$ is isosceles as $BA = CA = DA$

Question 6 **(6 marks)**

$OABC$ is a trapezium with $\mathbf{OC} = 2\mathbf{AB}$. M lies on the diagonal OB so that $\mathbf{OM} = \frac{1}{3}\mathbf{OB}$ and N lies on the diagonal CA so that $\mathbf{CN} = \frac{1}{3}\mathbf{CA}$. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$.

By determining a vector for $\mathbf{MN}$, or otherwise, prove that $ABNM$ is a parallelogram.



$\overrightarrow{OM} = \frac{1}{3}\underset{\sim}{b}$

$\overrightarrow{ON} = \overrightarrow{OC} + \overrightarrow{CN}$ $\quad \underset{\sim}{a} - \frac{2}{3}\underset{\sim}{b}$

$= 2(\underset{\sim}{b} - \underset{\sim}{a}) + \frac{1}{3}(\underset{\sim}{a} - 2(\underset{\sim}{b} - \underset{\sim}{a}))$

$= \frac{4}{3}\underset{\sim}{b} - \underset{\sim}{a}$

$\therefore \overrightarrow{MN} = \frac{4}{3}\underset{\sim}{b} - \underset{\sim}{a} - \frac{1}{3}\underset{\sim}{b}$

$= \underset{\sim}{b} - \underset{\sim}{a}$

$= \overrightarrow{AB}$

Hence ABMN is a //ogram

since AB // MN

& AB = MN

Question 7 **(6 marks)**

(a) Show that $\frac{x}{x+1} < \frac{x+1}{x+2}$ when $x = 1.5$ but not when $x = -1.5$. (2 marks)

$x = 1.5$

$LHS = \frac{1.5}{2.5} = \frac{3}{5} = \frac{21}{35}$

$RHS = \frac{2.5}{3.5} = \frac{5}{7} = \frac{25}{35}$

$\frac{21}{35} < \frac{25}{35}$

$x = -1.5$

$LHS = \frac{-1.5}{-0.5} = \frac{15}{5} = 3$

$RHS = \frac{-0.5}{0.5} = -1$

$\therefore 3 \not< -1$

(b) Prove by contradiction that, for every positive real number x, $\frac{x}{x+1} < \frac{x+1}{x+2}$. (5 marks)

assume $\frac{x}{x+1} \geq \frac{x+1}{x+2}$

$\frac{x}{x+1} - \frac{x+1}{x+2} \geq 0$

$\frac{x(x+2) - (x+1)^2}{(x+1)(x+2)} \geq 0$

$\frac{x^2 + 2x - x^2 - 2x - 1}{(x+1)(x+2)} \geq 0$

$\frac{-1}{(x+1)(x+2)} \geq 0$

this is a contradiction since $x > 0$ and $\frac{-1}{(x+1)(x+2)} < 0$

OR $x(x+2) \geq (x+1)^2$ as $x+1$ & $x+2 > 0$ since $x > 0$

ie $x^2 + 2x \geq x^2 + 2x + 1$

ie $0 \geq 1$ false $\therefore$ contradiction

End of questions